

THREE CONSTRUCTIONS OF THE APOLLONIUS CIRCLE OF A TRIANGLE A CIRCLE TANGENT TO THE THREE EXCIRCLES

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ABSTRACT. For a non-degenerate triangle ABC , the nine-point (Euler–Feuerbach) circle is tangent to the three excircles. We study another remarkable circle: the circle tangent to the three excircles in the Apollonius contact configuration. We present three complementary constructions: (i) a construction via notable lines (Brocard axis and the line NS), (ii) an inversion centered at the Spieker point (conceptual core), and (iii) a Tucker-hexagon realization. We also explain how these constructions describe the same circle.

1. Introduction

Let ABC be a non-degenerate triangle and let $\omega_a, \omega_b, \omega_c$ be its excircles (opposite A, B, C), with centers J_a, J_b, J_c . We consider the circle tangent (in the relevant contact configuration) to the three excircles and call it the *Apollonius circle of ABC* in this context.

The purpose of this note is to collect three practical ways to obtain this circle:

- (1) an alignment construction using the Brocard axis and the line through the nine-point center and the Spieker point;
- (2) an inversion construction centered at the Spieker point, mapping the nine-point circle to the desired circle;
- (3) a Tucker-circle viewpoint via a concyclic hexagon determined by antiparallel segments of length s (the semiperimeter).

2. Notation and classical centers

We use the following standard triangle centers:

- O : circumcenter;
- N : nine-point center;
- S : Spieker point (incenter of the medial triangle);
- L : Lemoine point (symmedian point). (In your Lua code this is often denoted by K .)

The line (OL) is the *Brocard axis*.

We denote by $\mathcal{N}$ the nine-point circle. It is tangent to each excircle at the Feuerbach points

$$F_a \in \mathcal{N} \cap \omega_a, \quad F_b \in \mathcal{N} \cap \omega_b, \quad F_c \in \mathcal{N} \cap \omega_c.$$

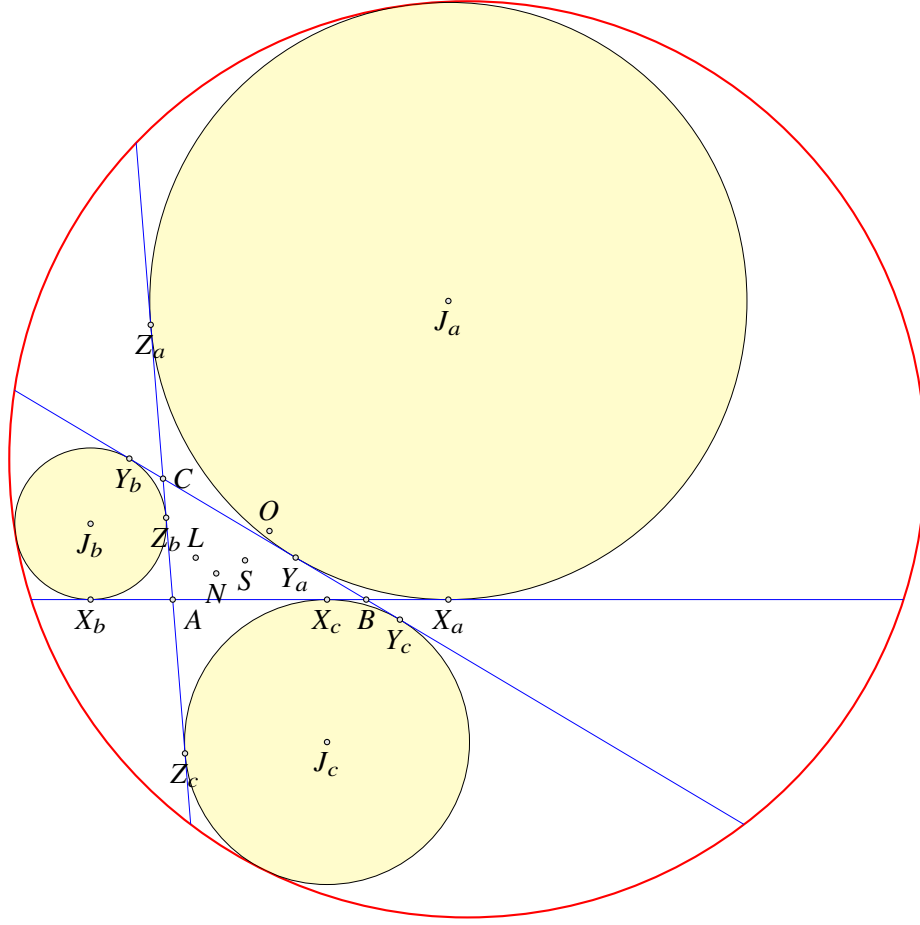


FIGURE 1. Starting point

3. Construction I: Brocard axis and Spieker line

- The *Brocard axis* of a triangle is the line through the circumcenter O and the symmedian (Lemoine) point L . [?, ?]
- The *Spieker point* S is the incenter of the medial triangle and, equivalently, the radical center of the three excircles. [?, ?]
- We can verify that S is the internal center of similarity of the Feuerbach circle (nine-point circle) and the Apollonius circle of the triangle. This implies that the points N , S , and Q are collinear. [?]
- We also recall that the nine-point circle N is tangent to the three excircles (Feuerbach's theorem). [?, ?]

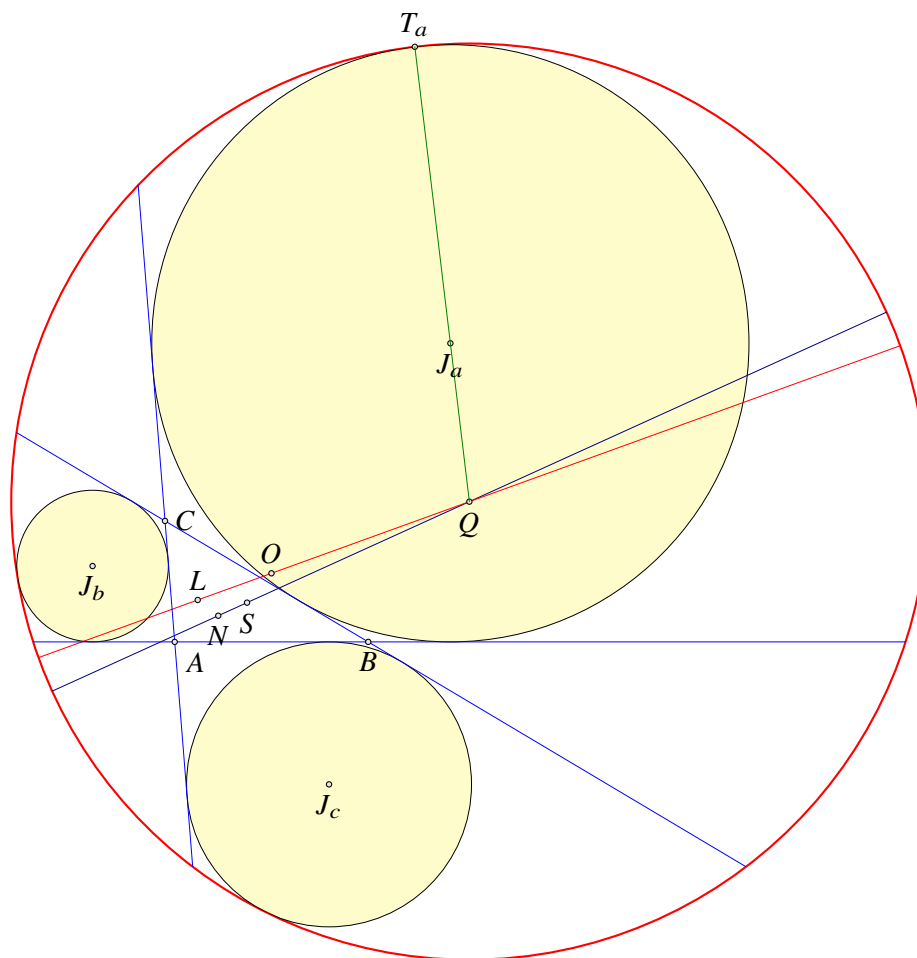
Theorem 1 (Brocard–Spieker construction). *Let ABC be a non-degenerate triangle. Let O be the circumcenter, L the Lemoine point, N the nine-point center, and S the Spieker point. Then the center Q of the Apollonius circle is obtained as*

$$Q = (OL) \cap (NS).$$

Remark. Theorem ?? is presented here as a practical construction rule. A conceptual explanation is given by the inversion approach of Section ??, where the Apollonius circle arises as the image of the nine-point circle under an inversion centered at S . [?]

3.1. Practical steps

- (1) Construct O and L and draw the Brocard axis (OL) . [?]
- (2) Construct N and S and draw the line (NS) . [?]
- (3) Set $Q = (OL) \cap (NS)$; this is the center of the Apollonius circle.

FIGURE 2. $Q = (OL) \cap (NS)$

4. Construction II: inversion at the Spieker point (core method)

4.1. Radical center and orthogonal circle

Let S be the Spieker point of ABC . It is the radical center of the three excircles:

$$\text{Pow}_{\omega_a}(S) = \text{Pow}_{\omega_b}(S) = \text{Pow}_{\omega_c}(S) = p > 0.$$

Hence the circle

$$\rho = \odot(S, \sqrt{p})$$

is orthogonal to each of $\omega_a, \omega_b, \omega_c$.

4.2. Inversion

Consider the inversion $\mathcal{I}$ with center S and radius $\sqrt{p}$. Since ρ is orthogonal to each excircle, the inversion leaves each excircle invariant. Therefore the image

$$\mathcal{A} = \mathcal{I}(\mathcal{N})$$

is tangent to $\omega_a, \omega_b, \omega_c$.

Definition 1. *The circle $\mathcal{A}$ is called the Apollonius circle of triangle ABC (in this contact configuration). It is tangent internally to the three excircles.*

If $T_a = \mathcal{I}(F_a)$, $T_b = \mathcal{I}(F_b)$, $T_c = \mathcal{I}(F_c)$, then T_a, T_b, T_c are the tangency points of $\mathcal{A}$ with $\omega_a, \omega_b, \omega_c$.

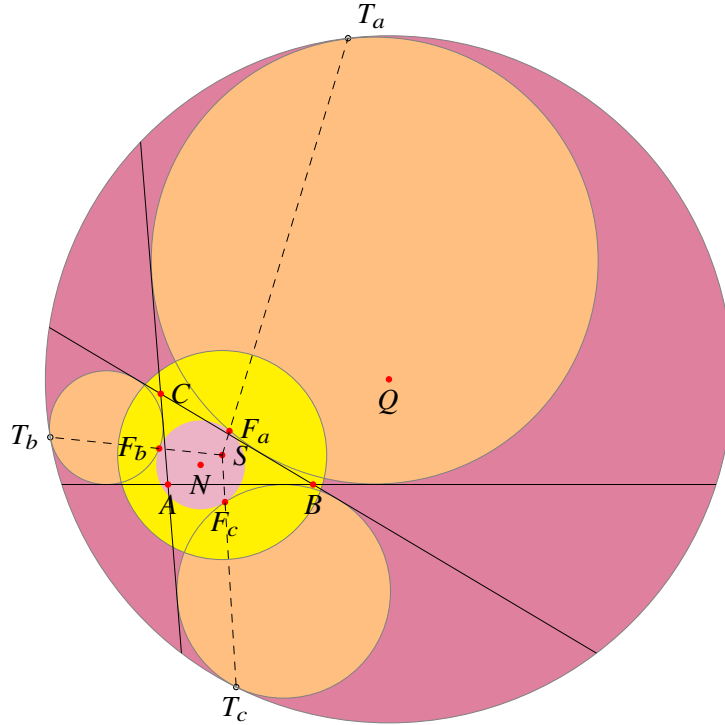


FIGURE 3. $\mathcal{A} = \mathcal{I}(\mathcal{N})$

5. Construction III: Tucker circle viewpoint

This section explains how a Tucker hexagon yields the same circle.

5.1. A preparatory lemma

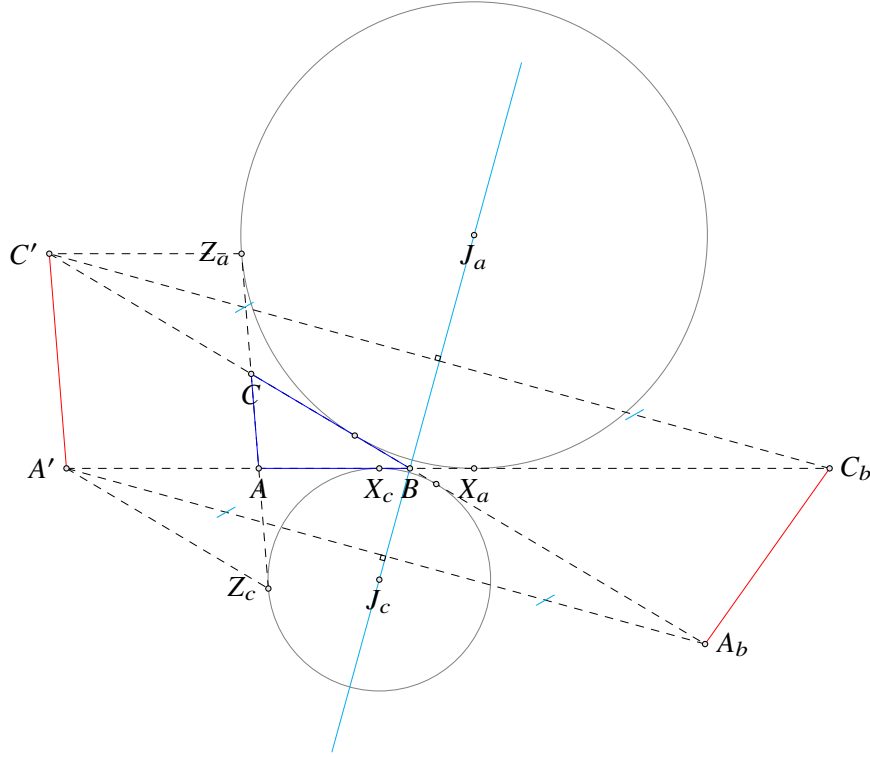


FIGURE 4. A chord antiparallel to one side of the triangle with length s

Lemma 1. *In the configuration defined by the first construction, the segment $[A_bC_b]$ has length s and is antiparallel to $[AC]$.*

Proof. One may use the equalities $AZ_a = AX_a$ and the perimeter relation $AZ_a + AX_a = AB + AC + BC = 2s$ to obtain $AZ_a = AX_a = s$, and then deduce the claimed parallelism and length relations. $\square$

5.2. Tucker hexagon and concyclicity

Applying the lemma to each pair of sides yields points

$$C_a, C_b \in AB, \quad A_b, A_c \in BC, \quad B_a, B_c \in CA$$

such that the segments

$$B_aC_a, \quad A_bC_b, \quad A_cB_c$$

are antiparallel to BC, CA, AB respectively and all have length s .

These six points form a Tucker hexagon and are concyclic; the resulting circle coincides with the Apollonius circle.

(See for a demonstration: D. Grinberg and P. Yiu, *The Apollonius Circle as a Tucker Circle*, Forum Geometricorum 2 (2002), 175–182.)

Proof. Let $a = BC$, $b = CA$, $c = AB$ and $s = \frac{a+b+c}{2}$. Assume the antiparallelism and $|B_a C_a| = |A_b C_b| = |A_c B_c| = s$ as above.

If $P, P' \in AB$, $Q, Q' \in BC$, $R, R' \in CA$, then the six points P, P', Q, Q', R, R' are concyclic if and only if

$$(1) \quad \frac{AP}{BP} \cdot \frac{AP'}{BP'} \cdot \frac{BQ}{CQ} \cdot \frac{BQ'}{CQ'} \cdot \frac{CR}{AR} \cdot \frac{CR'}{AR'} = 1,$$

where all segments are directed.

Since $B_a C_a$ is antiparallel to BC , we have $\triangle AC_a B_a \sim \triangle ABC$ and therefore

$$\frac{AC_a}{AB} = \frac{AB_a}{AC} = \frac{B_a C_a}{BC} = \frac{s}{a},$$

hence $AC_a = \frac{s}{a}c$ and $AB_a = \frac{s}{a}b$ and

$$\frac{AC_a}{BC_a} = \frac{s}{a-s}, \quad \frac{CB_a}{AB_a} = \frac{a-s}{s}.$$

Similarly,

$$\frac{AC_b}{BC_b} = \frac{b-s}{s}, \quad \frac{BA_b}{CA_b} = \frac{s}{b-s},$$

and

$$\frac{BA_c}{CA_c} = \frac{c-s}{s}, \quad \frac{CB_c}{AB_c} = \frac{s}{c-s}.$$

Substituting into (??), all factors cancel and the product is 1. Thus the six points are concyclic. $\square$

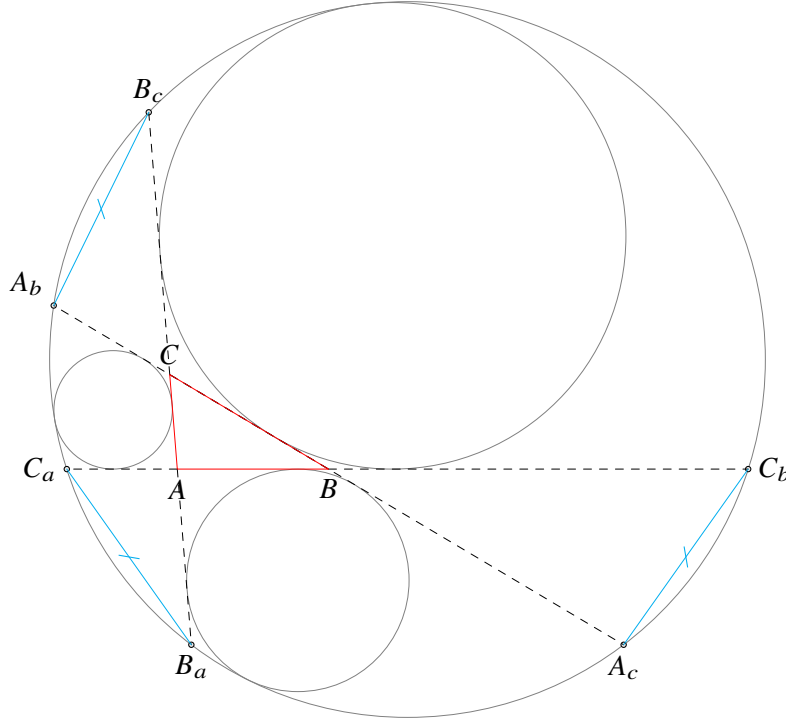


FIGURE 5. Apollonius circle is a Tucker circle

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